Chapter 3. Diagonalizing Over Infinity

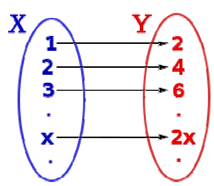
In Chapter 1 we briefly mentioned Georg Cantor's proof of the un-countability of the real set. This has a lot of bearing on the subject of continuity, which we have already touched on. The nature of the real set and continuity are essential to the way mathematics is done in a great deal of physics and has influenced an even greater portion of mathematics. Therefore, we'll begin our discussion by taking a closer look at what Cantor was exploring during the late 19th century.

Cantor began to study the nature of the infinite in mathematics. The physical world, while rich in multiplicity, does not seem to include infinitudes of anything physical. However, Mental Space appears to have no such problem, and some mathematicians have employed mathematics as a tool for studying "precise" ideas about infinity. Even the simple set of natural numbers ($\mathbb{N}$) is infinite. It is open-ended and you can just keep on counting "forever". We can say that the natural numbers form a well-ordered set that is precisely defined at its beginning (the number 1), but undefined (or we may say only vaguely defined) at its end. It has no well-defined end, because, for any natural number n you can always generate another number $n + 1$. Actually, the beginning of the natural number set is not so well defined either. Most people would start at 1, but some people prefer to start at 0. So we have to specify $\mathbb{N}_0$ or $\mathbb{N}_1$. The problem with 0 is that it is a "back formation". By this I mean that the numbers originated as a way of describing "how many", and that meant you were indicating how many examples of a particular item. When you speak of having "zero" items, you imply the prior condition of having had some of those items or the future possibility of having some of those items, still based on a prior condition of at least knowing of their existence somewhere. The notion of the existence of an item precedes the notion of not having the item. Thus it is odd to begin an enumeration system with a number that refers to not having any of something before you get to having anything in the first place.

In modern physics scientists have come to accept that everything exists potentially in the vacuum state. Nevertheless, we still have the problem of talking about a "lack" of something when we do not know what it is we lack, especially if we know that, whatever it is, it exists as a potentiality that we do not experience simply because our attention is not focused in the proper direction or manner in order to perceive that something. Thus, we find that there is an issue about using 0 as a general symbol for the lack of something when the something perhaps exists but remains undefined and/or unknown. This means that 0 remains an undefined variable unless it is defined by the prior knowledge of something to which it refers that can only happen after we have knowledge of at least 1 of the somethings. We can say, "I see one cow." We can also say, "I see no cows." However, it is nonsense to say, "I see no x's" without prior knowledge of some expressions (such as the word "expressions") containing x's. We are not conveying any information in such conditions, so 0 is not a proper beginning for the set of natural numbers. In my opinion, if we have to have a number zero, it serves better as an integer bridge to the negative integers. Zero and the negative integers all imply the prior existence of the positive integers -- i.e., the natural numbers ($\mathbb{N}$). The traditional set of integers ($\mathbb{Z}$) always contains zero, the natural counting numbers ($\mathbb{N}$), and $\mathbb{N}$'s "additive inverses" which are known as the negative integers. The negative integers (as well as 0) all imply the prior existence of the positive integers, which are the same as the natural numbers. So the integers are an expansion of the natural numbers.

Because the set of natural numbers has no defined ending, it is said to be "infinite" (in-finis = no end). Cantor wondered whether all infinite sets really were the same size. Before that everyone pretty well assumed that infinity (the condition of being infinite or undefined in some way) was the only infinity, and it did not come in different sizes. So Cantor developed some techniques for studying infinity. His first tool was simply to map sets one-to-one. He used as his "metric", the set of natural numbers (ℕ). He **assumed** that this would be the simplest form of infinity. Of course, this assumption is not exactly correct, because we just found out that the set of natural numbers starting with 1 is well defined at its "beginning". Zero, thought of as "nothing" somehow existing forever with no "before" or "after" existence of anything else, might be the simplest form of infinity. But even that thought implies the notion of "something else" (and the notion of definition) in order to conceive of it. So we will grant that Cantor was speaking here of something that has a beginning and goes on without "end".

What happens if we just take half of the natural numbers, say the even numbers. Do we get half an infinity? Cantor lined up the natural numbers with the even numbers and found that they mapped one-to-one.



Interesting. Half an infinity equals a whole infinity. Infinite sets apparently do not follow the ordinary rules of arithmetic.

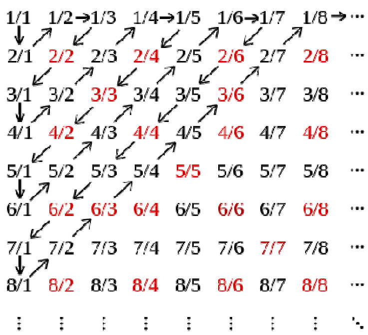
What about the integers (ℤ)? All the positive numbers plus all the negative numbers should equal two infinities, right? Wrong.

Cantor found that by folding the list in half he could map the positive and negative integers to the natural numbers.

* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,......
* 1, -1, 2, -2, 3, -3, 4, -4, 5, -5,.....

So two times infinity also equals infinity.

Cantor went on to look at the rational numbers (ℚ). Here there seemed to be a problem. The rationals **must** be more numerous than the natural numbers. They form a ratio of one infinity to another, so we have two infinities of natural numbers interacting: (m / n) so that there is an infinity of rationals between any two natural numbers. Cantor organized the rationals into a neat array. If you count forever down the first row of the array, you'll never even finish the first row. How can the rationals be countable? However, Cantor noticed that by counting diagonally over his array he could map the rationals to the natural numbers. (Follow the arrows marked on the array below and map them one to one to the natural numbers.)



Bingo!! He had counted the rational set ℚ, and it was the same size as ℕ. He had also found a creative way of counting that was orderly but did not follow an ordinal sequence $a < b$. If you insist on counting Cantor's array of rationals horizontally or vertically, then you CANNOT map the list to ℕ. If you allow diagonal counting, then ℚ maps to ℕ with no problem. It depends on how you look at it.

What about the real numbers (ℝ)? Cantor could organize set ℚ so that it could be written in an orderly (though not "ordinal") fashion. But nobody has yet been able to write down the entire set ℝ in a "well-ordered" ($a < b$) much less in any fashion, even though Zermelo apparently has shown that it is theoretically possible to do so. So the best Cantor could do was assume that there was some way to compose them into a list. Actually, the notion of listing ℝ in order is quite strange, because most of its members are non-algorithmically irrational (have no predictable pattern to their sequence of digits) and therefore can not be symbolized by any precisely written out number, which is why we use substitute symbols such as π or φ for important ones that are not simple algebraic numbers like $\sqrt{2}$.

Once Cantor had a "complete list" (a feat which he accomplished by simply declaring that he had one), he found that, by adapting his "diagonalizing" technique in another creative way, he could generate a number that was NOT on his list!! This seemed to contradict the assumption that he had already compiled a complete list. His conclusion was that there is no way to get a complete list of ℝ, and therefore ℝ is a "bigger" infinity than ℕ and therefore also must be uncountable (a condition also called "non-denumerable").

Let us take a look at Cantor's "proof". It's actually a kind of demonstration or construction that leads to a contradiction **if we accept his assumptions**. This mechanical demonstration aspect makes it particularly interesting from the standpoint of physics. Here is a version of Cantor's demonstration based on Eves and Newsom, p. 255-256. For a more detailed discussion, see

http://www.jamesrmeyer.com/infinite/diagonal-proof.html.

See also Meyer's kindle ebook, **The Infinity Delusion**. As the latter title indicates, Meyer is one of a group of mathematicians who have problems with Cantor's "proof".

Our purpose in looking at this demonstration is to play with Cantor's ingenious technology, not to invalidate the proof (although we will bring up some suspicious aspects of the proof, and you can find plenty more of that in James Meyer's "Logic and Language" website). We will use the binary number system for this version of the proof, so each digit "a_{ij}" in the proof will represent only the digits 0 or 1 exclusively.

Theorem: The set of all real numbers in the interval ($0<x<1$) is non-denumerable.

1. Assume the set is denumerable.
2. List the numbers in the sequence $\{P_1, P_2, P_3, ...\}$.
3. Each (P_i) can be represented uniquely as an infinite decimal.
4. Form the sequence of numbers into an array where j is the jth digit of the sequence of a_{ij} digits for the number P_i:

$$
\begin{array}{l}
P_1 = 0.a_{11}\ a_{12}\ a_{13} \ldots \\
P_2 = 0.a_{21}\ a_{22}\ a_{23} \ldots \\
P_3 = 0.a_{31}\ a_{32}\ a_{33} \ldots \\
\ldots\ldots\ldots
\end{array}
$$

5. We can construct a number $P_x = 0.b_1\ b_2\ b_3 \ldots$ in which $b_k = 0$ if $a_{kk} = 1$, and $b_k = 1$ if $a_{kk} = 0$, for $k = 1, 2, 3, ..., n, ...$ (The number described by the sequence of digits a_{kk} we will call the "diagonal" number, and the "constructed" number P_x we will call the "flipped diagonal" number. The digit a_{kk} is the digit where $i = j$.)
6. Such a number clearly lies between 0 and 1 and differs from (P_1) in the first decimal place, (P_2) in the second place, (P_3) in the third place, and so on, thus producing a new number that is not on the list.
7. Thus the original assumption is untenable, and the set is non-denumerable. QED!?

Here is what such a list might look like. Notice that we have no way of predicting what the sequence of 0's and 1's will be like once we arrive at the . . . series of dots. We only know that the diagonal sequence of numbers in red will flip to their opposite values and the resulting binary number in blue down below will be different from all numbers on the original list. In the list *S* with a subscript number stands for an infinitely long string of digits that in this presentation consists only of 0's and 1's. The subscript identifies which item on the list the string belongs to.

$$
\begin{array}{ll}
s_1 & = {\color{red}0}\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ldots \\
s_2 & = 1\,{\color{red}1}\,1\,1\,1\,1\,1\,1\,1\,1\,1 \ldots \\
s_3 & = 0\,1\,{\color{red}0}\,1\,0\,1\,0\,1\,0\,1\,0 \ldots \\
s_4 & = 1\,0\,1\,{\color{red}0}\,1\,0\,1\,0\,1\,0\,1 \ldots \\
s_5 & = 1\,1\,0\,1\,{\color{red}0}\,1\,1\,0\,1\,0\,1 \ldots \\
s_6 & = 0\,0\,1\,1\,0\,{\color{red}1}\,1\,0\,1\,1\,0 \ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,{\color{red}0}\,0\,1\,0\,0 \ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,{\color{red}1}\,0\,0\,1 \ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,{\color{red}1}\,1\,0 \ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,{\color{red}0}\,1 \ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,{\color{red}0} \ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\ \ddots
\end{array}
$$

$$
s = {\color{blue}1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1} \ldots
$$

Cantor's big assumption is that he can somehow create a complete list, even though he apparently can not figure out how to organize the set of real numbers in the precisely noted manner he does with the natural numbers, integers, or even rational numbers so that we can predict the exact value of each item on the list. Also he does not provide us with an algorithm for generating his list the way he does for the integers and the rationals. Nor does he offer any **proof** that the list is **complete** as he asserts. Thus his claim that his newly generated number is not on the list is questionable, because we have no way to really be sure his list is complete. Nevertheless we will begin by granting him that unproven assertion on which his proof of un-countability (non-denumerability) by "contradiction" rests and see what happens when we run some experiments on it to test the strength of his assertion.

Cantor gives us a demonstration that **looks** very much like a complete list with nice indexes that run in numerical order. Nevertheless it is probably a bogus list because only the indexes are orderly, and the list's internal content is totally lacking in algorithm. We have no way to inspect the list and know that it is complete other than accept Cantor's assertion that it is so. This is very different from his prior demonstrations with the even numbers, integers, and rationals. In fact, Cantor **never actually shows us a single real number on his list**. This is how magicians mislead people when they do illusions. If his list is really NOT complete, then the number that he generates by diagonalizing may indeed be definitely NOT on his list, but can be just a number that he somehow missed in his list because he did not have a systematic listing method in his description of the list, and thus it is not a new decimal. Another possibility is that for **some other unspoken reason** his diagonal flipped number somehow will always end up being already in his list, despite it apparently being different from each number on the list at some particular digit.

If you make the list without an algorithm or some kind of organizing rule you can never be sure it is complete since you have to keep track of an infinity of numbers! Cantor's labels in no way organize or identify the contents of his list. They simply produce a hypothetical array – contents unknown. Meyer would say that Cantor has constructed a **metalanguage** with which to describe his list, and the metalanguage has no clearly defined connection to the specific numbers on the list. A metalanguage is a language used to describe another language. Natural numbers in the metalanguage that describes the list may treat numbers in the lower level language that formulates the list of "real numbers" simply as objects with no numerical value! Cantor asserts that his list is complete, and he assumes that it is, and he uses indexes to convince you into believing him!! I think this hidden assumption about completeness is such an important part of his demonstration that **he must clearly prove or precisely define in his metalanguage (P_i) how his lower level language list comes to be a complete list of real numbers** before his non-denumerability proof may stand firm. Only then can we be sure that he gets a contradiction by finding another number that is not in his "complete" list.

Cantor was attacked in his day by members of the intuitionist school of mathematics such as Kronecker, Brouwer, and Poincaré (all of whom were mathematicians of the highest caliber) because his list was not truly constructible (as I just pointed out above).

Zermelo showed that the large transfinite numbers promoted by Cantor based on his "proof" that $\mathbb{R}$ is "bigger" than $\mathbb{N}$ are not derivable in principle from the standard Zermelo-Fraenkel (ZFC) axioms of mathematics. Many believe that Gödel showed that the continuum of real numbers cannot be proved or disproved (although Meyer and others also have doubts about the validity of Gödel's proof). You as the observer-participant must decide for yourself what to believe. So we shall experiment.

One way to make sure Cantor's list is complete is to provide an algorithm for checking the list's actual content, not merely its indexes – which are just a list of natural numbers. Of course, a finite algorithm is equivalent to a counting method (step 1, step 2, step 3, ...). So, by providing an algorithm, Cantor immediately would admit that the real numbers are countable after all. All Cantor does instead is shuffle indexes like a shell game. You don't know what is behind them. I call the indexed strings of digits dummy numbers. They are meaningless objects referred to in Cantor's metalanguage.

So, can we organize our list systematically to make sure we have them all and then test Cantor's "contradiction"? There are many ways to organize the list. Here is a very simple one. We just make a mirror image of the sequential enumeration of the binary natural numbers flipped to the right of the radix point (for partial values of unity represented as the addition of infinite sequences of decreasing fractions) and padded out to infinity with 0's from the point beyond which there are no more 1's. This mirror image list is not sequentially ordered by the dyadic relations < or > but has its own logical algorithmic sequence so that we can predict precisely from each entry what the next entry on the list will be. So here is our list that includes all binary numbers *x* such that $0<x<1$.

0.	[0.0000...]
1.	0.10000....
10.	0.010000...
11.	0.110000...
100.	0.0010000....
101.	0.10100000...
110.	0.011000000...
111.	0.1110000000...
1000.	0.00010000000...
1001.	0.100100000000...
1011.	0.1101000000000...
1100.	0.00110000000000...
1101.	0.101100000000000...
1110.	0.0111000000000000...
1111.	0.11110000000000000...
10000.	0.000010000000000000...
...	....
.....	....
....11111.	[0.111111111....]

This list includes every number from 0 to 1, right? The numbers in brackets represent 0 and 1, so they are not part of the list, but act sort of as bookends. Recall that 0.1111... = 1.00000.... and is therefore beyond the hypothetical "last number" on the list both in size and according to our algorithm. It is what we call the "limit" of the orderly sequence, and 1.000 is not to be included on the list by our current definition.

You may not agree. You may say that these numbers all have infinite 0 tails after them. There must be binaries that have 1's scattered through them all the way to infinity. In our list the 1's travel to the right slower than the 0's, but they both eventually get to infinity and cover all possibilities of sequences of 0 and 1. Because 0.111111..... is not allowed, we have a problem with the addition of an infinite sequence of smaller and smaller unitary fractions (fractions with numerator 1; fractions with numerator 0 represent empty digits in the sequence with 0 as a placeholder):

* 1/10 + 1/100 + 1/1000 + 1/10000 + 1/100000 + 1/1000000 + 1/10000000 . . . .

It's like a tortoise and hare race where the 0 hare seems to give the 1 turtle a little head start. The 0 hare races out way ahead infinitely fast, but the 1 turtle keeps plodding on and eventually "catches up" to the 0 hare. At infinity it turns out to be a tie, but the very "last" number on the infinite list is the tortoise number 0.111111…., so he wins by "cheating" and jumps out beyond the race track leaving the 0 hare all by himself. Another way of looking at it is to say that the non-local 0 hare is already at infinity waiting forever, while the 1 turtle plods his way across the local numbers until he gets to infinity too, although by definition there is no defined end of the sequence -- only a limit that is beyond the list but that also defines the list. Before getting into the "completeness" issue I want to play with the process of diagonalizing that unfolds with the particular infinite list that we have created according to Cantor's rules so that we can see a serious problem with our notation system for numbers when we start dealing with the addition of infinite sequences of decreasing fractions. The sequence of sums approaches a finite limit, but is not allowed to include the limit in its sequence of sums.

The list goes from 0.100... to as close to .111111111... as you please, but never quite gets there until you (by a transcendental leap) reach the limit of the list (1.0) at infinity. But that's OK, since it's an infinite list, presumably just like the natural numbers. For purposes of organizational clarity I'll leave the two bookend limits 0 and 1 there in brackets for reference.

So if we exclude both 0.11111... and 0.00000... from our list, leaving them there just as references, like we did when we wrote our definition of the list ($0<x<1$), where all the x's form the list, we get the following diagonal (with digits in red):

[0.0000...]
0.10000....
0.010000...
0.110000...
0.0010000....

0.10100000...
0.011000000...
0.1110000000...
0.00010000000...
.....
[0.111111111.....]

Our flipped diagonalized number becomes 0.001111111111....

However, according to our rules that require disallowing infinite strings of 1's, this decimal is thus equivalent to 0.010000...., which is obviously the second number on our **precisely defined** list!! Fortunately it's not way down in the list, though it could be, depending on how we organized things.

Our list seems to satisfy Cantor's criteria for such an infinite list. Whether it is a **complete** list of all of ℝ is another question. Every P_i in the set $\{0<P_i<1\}$ consists of an infinite string of a_{ij} digits to the right of the decimal that are either 0 or 1, and each P_i in the list is unique. We included every possible number of the pattern $\{0<P_i<1\}$ in our infinite list in an orderly fashion. And we diagonalized by his rules. We constructed a number:

* $(P_x) = 0.b_1\ b_2\ b_3 \ldots$ in which
* $(b_k) = 0$ if $a_{kk} = 1$, and
* $b_k = 1$ if $a_{kk} = 0$, for $k = 1, 2, 3, \ldots, n, \ldots$

Our number (P_x) differs from (P_1) in the first decimal place, from (P_2) in the second place, from (P_3) in the third place, and so on down the infinite list. Yet when we diagonalized, we didn't get a new number, we got one that is clearly already **on the list**. WHY????

There are two reasons.

First, there's the serious problem in our notational system that we mentioned earlier. In binary numbers this shows up as the overlap between numbers with infinite 0 tails and numbers with infinite 1 tails. We followed the standard rule to convert the infinite-1-tailed numbers into infinite-0-tailed numbers, but that just gave us a duplicate number (0.010000....) that was clearly already on the list.

Second, binary and base (n>1) numbers are more compact than unary (base n=1) "tally" numbers, but tally numbers do not have values smaller than unity, so Cantor is unable to make his list using unary numbers. In the way that we have organized the list, the 1's move out to the right as we go through the list more slowly than the diagonal digits move to the right as we go down the list. So, as we go to infinity, we diagonalize the whole infinite list in which each number is unique, but we always have a 0 at the diagonalizing digit after we have gone some finite distance down the list.

With natural numbers you can use mathematical induction to show that you have covered an entire infinite list. If (P_n) is some proposition defined for all natural numbers (n), and if (P_1) is true and [$P_{(k+1)}$] is also true, then (P_n) is true for the whole list of natural numbers, even though we have an infinite list. We can't use this procedure with Cantor's list as it is constructed because the indexes he uses do not refer to actual digits, and the numbers on his list are no more defined than the non-algorithmic irrational numbers that supposedly form the bulk of the set of real numbers.

However, since we have constructed our special list via an algorithm that precisely mirror images the binary natural numbers (adding infinite 0 tails on to all of them), we can use "mirror reflection" mathematical induction to cover the whole list. Our list of binary numbers between 0 and 1 is just a mirror map of the binary natural numbers that act as their labels. Our list is made totally of infinite sequences of binary digits and is infinitely long and systematically covers all possible combinations of 0 and 1 listed in a string of digits. The fact that right off the bat just by following Cantor's rules we get duplicates that are already ON the list is an interesting situation given Cantor's claims for his proof. It means his method as presented does not actually guarantee to generate a number that is not on the list in all infinite cases. What's more, since we have no way to observe all the numbers on the list, we have no way to check sure for sure whether a given diagonalized number is really on or not on a randomized list now that we have found a suspicious list that violates Cantor's logical argument.

It turns out we can construct an infinite number of these problem lists for Cantor. For example, take the list below, which is identical in content to the above list, but slightly rearranged:

[0.0000...]
0.010000....
0.100000...
0.110000...
0.0010010....
0.10100100...
0.011010000...
0.1110000000...
0.00010000110...
.....
.....
[0.111111111.....]

In this case I designed my list to provide some different special information about the contents of the list and its organization. Here we can have all sorts of binary sequences listed helter skelter except for one condition; the digits for a_{kk} are always 0. The diagonal becomes 0.000000000... and its flipped diagonal is then 0.111111.... Both of these numbers are well known and are **not** on the list. They are 0 and 1. But they are outside the range of our defined set and so **they do not qualify as additional members of the list that have not been accounted for**. So this is a different kind of unexpected outcome. Here's another interesting case:

[0.0000...]
0.10101010....
0.1000000...
0.011000...
0.0010000....
0.10101000...
0.011000000...
0.1110001000...
0.00010000000...
.....
.....
[0.111111111.....]

In this next example I deliberately made the diagonal match the first number on the list, which is 0.101010101010.... All numbers on the list after the first number can be helter skelter except that the diagonal digit a_{kk} must match the value of digit a_{1j} where j is the column in which a_{kk} appears. This base 2 number corresponds to the base 10 value 0.6666666.... It also corresponds to the fraction 2/3 in base 10. When you flip the binary diagonal you get the sequence 0.010101010101...., which corresponds to the decimal 0.3333333...., which represents the fraction 1/3 in base 10. This is an interesting case, because the flipped binary turns out to be a number so common that it would certainly be on any "complete" list of binaries. We can call this number the "limit" of the list, because the diagonal is unknown in entirety until we get "all the way" through the infinite list even though we have programmed it so that we know the entire sequence **in advance** because we designed an algorithm that generates enough order in the list that we can predict our diagonals and flipped diagonals even though the rest of the digits in the list are distributed randomly.

We can now add another interesting rule to make sure our initial list is **complete**. We say that the list must contain every complement (or conjugate) pair of infinite binary sequences. A complement binary sequence is one in which every 0 is replaced by a 1 and every 1 is replaced by a 0. In other words a complement binary sequence is the flipped version of any given binary sequence and of course means that the flipped diagonal **must be included in a complete list by definition**. In this way we include every possible flipped diagonal into the list. We know that if we have two lists, one being the "original" low level list or reals defined by Cantor in his metalanguage, and a second list being the same low level list plus the flipped diagonal from the first list, we can count the two lists as one list simply by placing them next to each other (List A being the original list, and List B being the original list, with the flipped diagonal placed at the head of List B and labeled item #0, and then matching their natural number metalanguage labels one-to-one.

A: 1, 2, 3, 4, 5, 6, ...
B: 0, 1, 2, 3, 4, 5, ...

We end up with our two popular versions of the natural numbers, $\mathbb{N}_0$ and $\mathbb{N}_1$. This procedure holds for any arbitrary diagonal we may define and its complement flipped diagonal. So we see here that by giving a precise definition of the completeness of our list and by specifying an arbitrary value for the diagonal, we end up with two infinite lists that are always equivalent according to Cantor's own system of one-to-one mapping that he uses on the natural numbers. We just count the two lists side by side. We can iterate the process as many times as we like adding an infinite number of new diagonals and new flipped diagonals, but each new list maps one-to-one to our original list or to any intermediate list.

Cantor's diagonal rule means that the flipped sequence always will be different from each sequence on the list at its a_{kk} digit. This means that Cantor may not impose his diagonal rule on a "complete" infinite list of binaries, because a complete set of binaries by definition contains every complementary pair of binary digit sequences. Doing so by definition violates our precise definition of a complete list of binary numbers which Cantor failed to provide. If his list is complete, it must contain every binary sequence. That must include every possible diagonal and its complement. However, by changing every item on the list by a unique digit he appears to contradict the completeness of the list by producing a new sequence that is not on the list. When he flips the diagonal into its complement, by our logical definition he generates a sequence that **must be on another list that is of the same order of infinity and therefore no "bigger" than the list of natural numbers.** Cantor's problem is that he starts with an infinite list of natural numbers, maps his real numbers to that list, and then generates a "new" real number that is not on his first list of real numbers, but along the way he forgets that by the rule for generating the natural numbers, given any n that is on the list, $n + 1$ is **always** still on the same list. He is treating an infinite list as if it were a finite list. As the natural numbers are defined (and as his list is defined by those natural number labels) adding another item to the list does not make a "bigger" infinity, it just adds another specified number to the infinite list which is how the natural numbers are defined. The list is just a list of unspecified length that enumerates a bunch of objects of unspecified or only vaguely specified semantic contents. You can see here how the metalanguage of his list treats the numbers on his list simply as an endless bunch of objects that are by definition governed by his metalanguage definition of the counting numbers and not by any property of the real numbers. So Cantor's method of diagonalizing real numbers is just a variation of the same way he diagonalized the rational numbers. He just adds another step, a diagonal list of lists of real numbers, and the list of lists can be mapped back to the natural number list labels.

When we look at what Cantor has really created, we see that he has contradicted his own fundamental system of one-to-one mapping with which he began his research into infinity. This unannounced shift of linguistic systems (rules of the game) for discussing ideas or phenomena is a serious problem in modern mathematics and physics and leads to many illogical conclusions that mislead even the experts. We now have shelf loads of books and numerous faculty positions dedicated to the study of transfinite numbers, all based on a simple linguistic misunderstanding mixed together with assumptions that contradict earlier foundational assumptions. This problem relates back to the difficulties mathematicians have had in framing a precise definition of the calculus by means of limits.

If we wish to evaluate Cantor's contribution to mathematics, we can say that he brought to our attention a critical flaw in our notation system for representing values that are less than unity. It turns out that no matter what base we use, many ordinary rational numbers can only be approximated in the notation that uses radix points. Likewise, many other numbers (a majority of the possible values less than unity) can only be approximated in that same notation. There always has to be a resolution cutoff point that means any value is only accurate to a certain agreed upon significant figure or number of decimal places. This becomes extremely important when mathematics is applied to real-world problems in physics, especially since there are fundamental limitations on the measurement of any physical system.

Another way of looking at the "contradiction" brought out by Cantor's "proof" is to treat the flipped diagonal as the "limit" of the list. If we return to our discussion of the way mathematicians defined a limit in order to justify the calculus, we find that Cantor's flipped diagonal "limit" corresponds to that method, since each infinite binary sequence after the radix point corresponds to the sum of a sequence of increasingly smaller fractions, and each succeeding digit of the diagonal to the list takes us to a smaller component fraction of a particular sequence. Cauchy's limit of a function (see Chapter 01) is a curious entity, because it is part of the set of values for points along the segment of a function but is a point specifically **excluded** from the series of points in the arbitrary segment that consists of an infinite set of real number valued points (just like Cantor's list). Recall that $|x - a|$ must stay greater than 0, which means that x may not become equal to a. So f_a becomes the "limit" but is not allowed to be treated in the function even though it is part of the set of points along the interval. In the same way the sequence 0.0101010101 is a member of the set of binary sequences even though it is excluded from the sequence that makes up the "list". We make a quantum leap to the limit and find that the limit point along the segment of the function is part of the list of points along the segment, and in fact is the one point we put attention on to find the derivative. No one would say that the limit point is not a part of the function since it must be a "continuous" function. On the other hand, the leap to the limit from the series of sums that converges on the limit is not a smooth transition, but a sudden shift of attention to a separate subset governed by a separate rule.

It is well known that the infinite number of points on a line of any length is the same as the infinite number of points in a space of any size or dimension. The addition of one point, or any finite number of points, or even an infinite number of points to an infinite collection of points does not make the infinite collection infinitely larger -- as Cantor himself showed by his one-to-one mapping. However, in his "diagonal proof of the non-denumerability of the reals" he abandons his one-to-one correspondence while at the same time failing to prove the completeness of his list. He leaves us only with a dummy list labeled with natural numbers in a metalanguage, and we know that any such list with another "new" natural number added is still of the same "infinite cardinality" as the natural numbers, because for any n we can always have $n + 1$. He does not write out his list or any precise rule for such a list, because, if he did so, he would merely have another list that maps to the natural numbers, which is not what he wants to show, because then his vision of a vast hierarchy of transfinite sets fades into a fantasy.

Now let's start to think about this little mathematical game in terms of physics.

Cantor's dummy indexed decimals don't really let you check his list, because you do not know the values of his (a_{ij})s. They are just probabilities. Each digit can be a 0 or a 1 with 50% (or .5) probability. This is just like the quantum problem of the observer not knowing whether the electron spin is up or down, a photon is polarized this way or that, or the Schrödinger cat is dead or alive. You must look at the number or the particle to see what the situation really is.

The quantum wave function is like Cantor's dummy indexed "number" system. A particle is like an actual number. When you produce an actual list, such as we did in our little experiment above, and look at actual numbers, then you know which way each digit is, 0 or 1 in each position. In the case above, where we can check the list, we find that the diagonalized number is indeed on the list or is the limit to the list (generated by the diagonal flip the way we slide a value in a function in a certain definite converging manner toward the limit of a defined range. This **substantiates** our claim that our list is indeed complete whether or not it has anything to do with $\mathbb{R}$. And it is not possible to construct a new number that is not on the list (the same way that the limit of our derivative is a number that is also in the set of points that are described by the range of the function that we choose to study -- a precise one-to-one map of the points on a line to the set of real numbers). The limit is in that set but is deliberately not included in the sequence of values that we define in order to "trap" the limit of the sequence. When we "collapse the wave function" by actually observing the flipped number from a real list, it falls into place somewhere on the list just like an observed particle appears somewhere in the range of its continuous real-value wave function (either within the list or as its unique "limit" that is found in an equivalent "meta-list"). Cantor never actually produced a single number on his list or a single actual diagonalized decimal or flipped diagonal like we just did in our several examples. Before we look at a flipped diagonal, this dummy diagonal made of a string of indexes hovers somewhere in a transcendental land of all possibilities outside the dummy list that we haven't really examined with our attention. This is what I mean by Observer Physics (and Observer Math.) Whether you actually look or don't actually look at something makes a world of difference. Is Schroedinger's cat alive or dead? Take a look.

If Cantor's diagonalization ever does produce one (and only one) extra number not on the list I suggested, it forms the "limit" of the range of the list. Multiple repetitions of diagonalizing give the same results. (Program: Put the new flipped diagonal number at the beginning of a new list. Diagonalize. Flip. Put the new flipped diagonal number at the beginning of a new list, diagonalize, flip, ...) Each new list adds one new number, but has no effect on the cardinality of the list.

Cantor did not put his list in ascending or descending sequence by value, but we created some examples that partially or completely organize the list into a kind of wave function. The sequence of items on his list of reals, if it is indeed complete as we showed it can be by adding a simple rule to describe the list, and REGARDLESS OF THE INDIVIDUAL NUMERICAL VALUES of the real numbers on the list (which do not matter here), provides a ONE-TO-ONE MAPPING to the points in a line. Any two lines, regardless of size or location, are equivalent topologically as sets of points. The numerical values are irrelevant in Cantor's dummy system that is not ordered. In a sense they are ordered by their natural number **labels**. It is just a set of points in an infinite digitized array. The extra number that seems to pop out from diagonal flipping is just the end point of a line segment or whatever point we designate as the limit of a converging process. Forget its numerical value. (Forget the location of the particle in space/time.) It's like flipping a coin. Careful study of this phenomenon reveals the secrets underlying the theory of limits and the calculus as well as insights into how quantum mechanics works.

This little exploration we carried out has a great deal to do with the way we manage our attention. Every time we shift our attention we generate a quantum leap in consciousness from one viewpoint to another. Such leaps are quantum mechanical processes that hold in mathematics and in physics. The problem is that in pure mathematics we imagine that we can arbitrarily set the level of resolution and the "significant figures" with which we represent a value, whereas in physics we must tailor our mathematical models and our numerical representations to the scale of accuracy with which we can measure the phenomena we are studying and modeling. Mathematical space can be perfectly fractal, but physical space is only quasi-fractal and shifts the rules of the game at different scales.

Going back to our earlier discussion of dots and gaps, Cantor's list of real numbers is like a sequence of dots, each dot with a companion gap, the space between any two numbers on the list. By virtue of being a list, by definition, the set is denumerable. It is nonsense to talk of a non-denumerable list and uncountable numbers. Numbers are for counting. To me an "uncountable" number suggests a variable, like (x). You cannot count it in a sequence, because it represents a range of values. So you do not "count" x's unless you start from a particular value for x and then assign a counting system. The space assigned to a gap is variable. But not the gap itself. It belongs to the category of non-count nouns, like air and water. The number of gaps in the list equals the number of dots unless we add one more dot to represent the limit -- the end of the line. That's what the flipped diagonal represents -- an extra dot that terminates the set.

This discussion also can lead us into the interesting territory of Zermelo's Well-ordering Theorem and its equivalent, the Axiom of Choice. No one has been able to well-order the above list of real numbers between 0 and 1 or similar sets -- i.e., produce an algorithm to put them in ($a < b$) order. You will notice that my sample lists above contain certain aspects of order, but are NOT well ordered. This is a nice exercise for someone. Using binaries, well order the infinite list of numbers between 0 and 1. Zermelo apparently proved it can be done!! (Hint: To do it we first have to resolve the notational problem of repeating 1 tails that interferes with constructing a good algorithm.) However, Cantor seems to process his entire infinitely long diagonal in one transfinite step to get his flipped diagonal, and that looks to me like he is exercising the Axiom of Choice -- a rule that allows one to make an infinite number of choices all at once. In our experiment we allowed that way of getting a diagonal and a flipped diagonal.

When I comment on Cantor's work, I in no way mean to detract from his genius as a mathematician. His diagonal system is a profound and creative invention, a truly simple and powerful technology. It is a very useful tool for deliberately shifting of viewpoints, though not necessarily in the way he intended it. Unexpected twists of evolution often happen during the process of discovery or invention. Cantor's work embodies the fundamental principle elucidated by Maharishi in his Science of Creative Intelligence that any truly powerful technology must be capable of transcending itself. The diagonal technique itself is an excellent model of a system transcending itself that can be applied in various ways. It is simple, elegant, natural, and robust -- all nice qualities for a scientific model. However, we must be aware of what we are doing when we make such transcendental leaps of attention and not take leave of our link with logic and reality.

As an analogy we can compare Cantor's diagonalizing process to the process of meditation that was promoted by the Maharishi. Consider each number on the list to be a thought in the mind of a person. Each succeeding thought is like the next number on the list. As the person moves down the list doing his "diagonal" meditation, he is moving in the direction of the infinite and the infinitesimal at the same time. At each succeeding number his attention shifts to subtler and subtler a_{kk}'s -- aspects of a given type of thought -- represented in the number by the smaller and smaller, finer and finer digits (fractions) of the number. When the meditator gets to the "end" of the list, his attention transcends the list and finds itself OUTSIDE the list. When he flips, he is not on any known list number. Any number far down on the list has to have as its diagonal a fraction with an extremely large, but not infinite denominator. At the "end" when he transcends the list, he is at "infinity". His attention becomes completely undefined (with a value of 0), and suddenly a new flipped diagonal number appears. This number is not on the list as he has known it. It may be a new creative thought that arises from the source of thought, or "source of numbers", beyond the list. Or it may morph into an ordinary number on the list, a regular everyday thought. This "new" but perhaps not "new" number then incorporates itself into the list either as a number already known or as a number not previously encountered in the list. However, this does not make the list any bigger, because infinity plus 1 is still just infinity. However often we repeat this process of meditating, transcending, and then coming out onto another thought, we have not generated a higher cardinal infinity, just an greater awareness of diversity within wholeness. By virtue of our prior definition of wholeness and completeness the "new" number by definition integrates fully into the wholeness. Whatever the diagonal is, the flipped diagonal is always its perfect complement.

Cantor devised a truly ingenious technology, don't you think? And it is intimately related to the calculus and other mathematical tools that are essential to modern physics.

As my lists showed, Cantor's system really does produce numbers that are NOT on his list. All the examples I gave did so. And, until we introduced our definition of completeness, we did not guarantee that any one of our lists was a complete set of $\mathbb{R}$, only that each was infinite. However, in each case with our lists the "new" number would always transform itself magically into a number that was already in the list or to the limit of the list which is also part of its inherent wholeness and not a greater infinity. This is like the mind of the meditating person automatically integrating itself back into ordinary life again after his transcendental experience during meditation, but with a greater sense of wholeness.

Before we leave this chapter let us explore the "analogy" I mentioned between numbers and quantum mechanics a bit more. Math provides wonderful models for looking at the world. As we commonly see with computers, numbers can be interpreted in many ways. One of the interpretations is graphical. Numbers can form bitmap graphics.

Infinite decimals are like infinite digital wave forms.

As we shift from the radix point to the right in any real number, we can imagine we are moving in space/time by orders of magnitude farther and farther away from a starting point. Things seem to get smaller as they get "farther away". Or we can imagine that we are zooming in to finer and finer levels of magnification of an object. Objects seem to get larger and larger as we zoom in. So if our attention is on objects, "zooming out" makes objects look smaller and "zooming in" makes objects look bigger. However, if our attention is on space, then zooming out makes the space we are aware of appear bigger, and zooming in makes the space we are aware of appear smaller. This is true of course only if we have some objects as reference that we assume are reasonably holding "still". Cantor's list is a reference frame, and his indexes allow us to locate ourselves anywhere in the space enclosed by this reference frame.

As you might guess, the subject of reference frames is quite important in Observer Physics. OP is a more general viewpoint than conventional physics, but conventional physics operates nicely as a subset of OP although some times the interpretation may be somewhat different.

The same sort of thing happens with time frames. Size is an illusion that is relative to the Observer and the reference frames he selects. Measurement is a mapping of two arbitrary systems that may be in the same frame or different frames of observation. The Observer does all the "zooming" with his attention. We do zooming in, zooming out, shifting, panning, focusing, defocusing, dividing, integrating, fixating, de-fixating, plus a few other tricks with our attention. Attention management is what physics and living is all about. Magicians know how to manage your attention for you so that you start to believe in miracles -- things happening that seem to transcend the laws of physics. Underlying physics is the scientific study of consciousness, awareness, viewpoint, attention, definition, belief structures, and the role of the Observer as witness and/or participant. Palmer's **ReSurfacing** is a handbook of attention management that is well worth exploring in this respect.

Exercise: Do #18 "Viewpoints" and then #19 "This and That" in **ReSurfacing.**

As we saw with the rationals and also the real numbers, these sets are countable or uncountable depending on how you look at them -- your choice of viewpoint --, and the **observer** decides both the rules of the game and the outcome. We may accept some given rules and then decide what is a possible outcome given those rules. Or we may decide on the outcome and then fill in the rules of the game and the steps of actualization along the way. Skill in the second approach leads to designer reality and the science of the future.

We can use OBSERVER PHYSICS and OBSERVER MATH to clarify subtle and profound issues in the foundations of science. The OBSERVER is critical to such processes, even in pure math. Maybe love and other methods of expanding consciousness provide ways by which people may find soul mates even when they seem non-locally separated.